Physics 523, General Relativity

Homework 8

Due Wednesday, 13th December 2006

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Problem 1

Consider a flat FRW universe, governed by the metric

$$ds^2 = -dt^2 + a^2(t)\left(dx^2 + dy^2 + dz^2\right), \tag{1.1}$$

filled with only relativistic material and a cosmological constant Λ; say the space has a big bang at coordinate time $t = 0$. We are to calculate the cosmic scale factor $a^2(t)$ up to an overall normalization and describe the asymptotic motion of photon travelling along the positive $\hat{x}$-axis.

Let us begin by quickly reviewing the Einstein equations for this universe[1]. We note that the components of the Ricci tensor and scalar curvature for this metric are

$$R_{tt} = 3\frac{\ddot{a}}{a}, \qquad R_{ij} = -\delta_{ij}\left(a\dot{a} + 2\dot{a}^2\right), \quad \text{and} \quad R = -6\left(\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2}\right). \tag{1.2}$$

And recall that a (single-component) perfect fluid with equation of state $p = w\rho$ has a stress energy tensor given by

$$T^a{}_b = \rho \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & w & 0 & 0 \\ 0 & 0 & w & 0 \\ 0 & 0 & 0 & w \end{pmatrix}. \tag{1.3}$$

The Einstein field equations are then

$$R_{ab} - \frac{1}{2}g_{ab}R = -8\pi G\left(T_{ab} - \frac{\Lambda}{8\pi G}g_{ab}\right), \tag{1.4}$$

where Λ is the cosmological constant[2]. Writing out the 'tt' Einstein equation, we find

$$3\frac{\ddot{a}}{a} - 3\left(\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2}\right) = -3\frac{\dot{a}^2}{a^2} = -8\pi G\left(\rho + \frac{\Lambda}{8\pi G}\right), \tag{1.5}$$

which implies

$$\therefore \dot{a}^2 = \frac{8\pi G}{3}a^2\left(\rho + \frac{\Lambda}{8\pi G}\right). \tag{1.6}$$

It turns out that we won't actually need any of the other Einstein equations.

The last equation we need relates the energy density to the cosmic scale factor. This comes about from the conservation of energy[3],

$$\frac{d}{da}\left(\rho a^3\right) = -3pa^2. \tag{1.7}$$

This equation is implied by the divergencelessness of T_{ab}, which is itself just a restatement of the Bianchi identity for G_{ab}.

At any rate, we can use the conservation of energy for a fluid with equation of state $p = w\rho$ to determine how ρ varies as a function of $a(t)$. We see

$$\frac{d}{da}\left(\rho a^3\right) = 3a^2\rho + a^3\frac{d\rho}{da} = -3w\rho a^2,$$

$$\implies \frac{d\rho}{da} = -3(1+w)a^{-1}\rho.$$

Solving this equation by simple integration, we have

$$\log(\rho) = -3(1+w)\log(a) + \text{const.} \qquad \implies \qquad \rho \propto a^{-3(1+w)}. \tag{1.8}$$

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[1]Because we are more familiar with the notation used by Weinberg—despite its oddities—our derivations will follow his. *However*, we will use $a(t)$ to denote the cosmic scale factor so as to avoid confusion with R.

[2]It is quite common to see Λ defined with the $8\pi G$ absorbed into its definition. We prefer to keep it structurally more similar to the metric than the stress-energy (which follows from the paradigm that Λ is a metric parameter as opposed to a vacuum energy).

[3]That this is a statement of the conservation of energy can be understood as follows: the amount of energy in a comoving box of size a^3 is just ρa^3; because we consider only a perfect fluid, as the box expands, the only leakage arises from the 'pressure' at the sides of the box—which has surface area $6a^2$. However, only half of this is lost because only half of the pressure along the faces of the box is due to 'outgoing' flow; so there is a net loss of $3pa^2$ worth of energy.

Using the fact that a relativistic fluid has an equation of state $w = \frac{1}{3}$, we observe that

$$\rho = \frac{3\beta^2}{8\pi G} a^{-4}, \tag{1.9}$$

where β^2 is a constant of integration.

Putting this into the Einstein equation (1.6),

$$\dot{a}^2 = \frac{8\pi G}{3} a^2 \left(\frac{3\beta^2}{8\pi G} a^{-4} + \frac{\Lambda}{8\pi G} \right) = \frac{1}{a^2} \left(\beta^2 + \frac{\Lambda}{3} a^4 \right). \tag{1.10}$$

This ordinary differential equation can be integrated directly[4].

$$t = \int_0^t dt = \int_0^a \frac{a'\, da'}{\sqrt{\beta^2 + \frac{\Lambda}{3} a'^4}} = \sqrt{\frac{3}{4\Lambda}} \text{arcsinh} \left(\frac{a^2}{\beta} \sqrt{\frac{\Lambda}{3}} \right);$$

$$\therefore a^2(t) = \beta \sqrt{\frac{3}{\Lambda}} \sinh \left(2t \sqrt{\frac{\Lambda}{3}} \right). \tag{1.11}$$

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Let us now check that our solution agrees with the required boundary conditions. First, $a(t = 0) = 0$, as required; this shows that we were not unjustified in our organization of constants of integration when solving the differential equations above. Also, at very early times or when Λ is very small,

$$a^2(t) = \beta \sqrt{\frac{3}{\Lambda}} \sinh \left(2t \sqrt{\frac{\Lambda}{3}} \right) \approx 2\beta t, \qquad \text{for } t \to 0, \tag{1.12}$$

which is precisely what we would have obtained if setting $\Lambda = 0$ in (1.6). Similarly, in late times

$$a^2(t) \approx \beta \sqrt{\frac{3}{4\Lambda}} e^{2t\sqrt{\Lambda/3}}, \qquad \text{for } t \to \infty, \tag{1.13}$$

which is what we would have obtained if we had neglected the radiation density ρ altogether in equation (1.6).

Now, the motion of a photon in this space is entirely controlled by the condition that its worldline is null. For motion along the $\hat{x}$-axis, this is simply the statement that

$$0 = -dt^2 + a^2(t) dx^2, \qquad \Longrightarrow \qquad dx = \frac{dt}{a(t)}. \tag{1.14}$$

Again, this can be integrated—at least formally—so that if motion starts at the origin at time $t = 0$ then

$$x(t) = \int_0^t \frac{dt'}{a(t')}. \tag{1.15}$$

Although this integral can be done analytically in terms of hypergeometric functions—(what can't?)—it is far from illuminating. Therefore, rather than computing the light trajectory $x(t)$ analytically for a generic two-component universe, let us analyze its motion in the asymptotic regions of interest.

We showed above that for very early times,

$$a(t) \approx \sqrt{2\beta t}, \qquad \Longrightarrow \qquad x(t) = \sqrt{\frac{2}{\beta}} \sqrt{t}. \tag{1.16}$$

Comparing this with the notation of the problem set, we have

$$\sqrt{\frac{2}{\beta}} = \left(\frac{3}{8\pi G B} \right)^{1/4} \qquad \Longrightarrow \qquad B = \frac{3\beta^2}{8\pi G}; \tag{1.17}$$

so the problem set's B is such that $\rho = Ba^{-4}$—which we could have guessed.

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[4] Several horrendous integrals appear in this problem set; most were solved with the aide of a computer algebra package.

Alternatively, for late times we should use the approximation (1.13) which gives

$$\Delta x = \left(\frac{4\Lambda}{3\beta^2}\right)^{1/4} \int_{t_i}^{t_i+\Delta t} dt'\, e^{-t'\sqrt{\Lambda/3}} = \left(\frac{12}{\beta^2\Lambda}\right)^{1/4} e^{-t_i\sqrt{\Lambda/3}}\left(1 - e^{-\Delta t\sqrt{\Lambda/3}}\right). \tag{1.18}$$

This implies that at late times the photon will essentially freeze it's position—advancing exponentially slower and slower as coordinate time goes to infinity. Indeed, if t_i is a time late enough[5] for the universe to be virtually dominated by Λ, then within the infinitude of time to the end of the universe, the photon will travel only the finite distance

$$x(\infty) - x(t_i) = \left(\frac{12}{\beta^2\Lambda}\right) e^{-t_i\sqrt{\Lambda/3}}. \tag{1.19}$$

Problem 2

We are to study a closed FRW universe which is 'radiation dominated for only a negligibly short fraction of its life' and determine how many times a photon released at the big bang can encircle the universe before the big crunch. Although it is quite likely that the author of the problem had a mostly-matter-dominant universe in mind, there are certainly other ways of interpreting the problem[6]. We will consider here only the most obvious interpretation of the problem—the one of a universe with matter and relativistic energy components.

Unfortunately, our analysis in problem 1 above was not sufficiently general to consider a closed universe with the metric

$$ds^2 = -dt^2 + a^2(t)\left\{\frac{dr^2}{1-kr^2} + r^2 d\theta^2 + r^2\sin^2(\theta)d\varphi^2\right\}, \tag{2.1}$$

where $k = 1$ for a closed universe. Therefore, we will need to quickly generalize that discussion to include $k \neq 0$.

Notice that the coordinate 'r' here is not a radius in the sense of a usual spherical geometry: by setting $k = 1$ we are forced to restrict r to the range $r \in [-1, 1]$—it is an angular coordinate. Indeed, that $k = 1$ describes the geometry of a three-sphere is made manifest by the change of variables $r = \sin(\lambda)$ so that the metric becomes

$$ds^2 = -dt^2 + a^2(t)\left(d\lambda^2 + \sin^2(\lambda)d\theta^2 + \sin^2(\lambda)\sin^2(\theta)d\varphi^2\right), \tag{2.2}$$

which by inspection is the metric of a three-sphere with fixed radius $a(t)$.

The only reason why we so digress is to clarify that fixed-λ and fixed-θ trajectories are only geodesics when $\lambda = \theta = \pi/2$—otherwise the orbit will not describe a great-circle on the sphere. The moral is that if we would like to study simple photon geodesics in a closed spacetime, we must set $\lambda = \theta = \pi/2$—or, equivalently, we must set the coordinate $r \to 1$.

Now, let us return to the metric (2.1) and derive the Einstein field equations. If we write the metric in the form

$$ds^2 = -dt^2 + a^2(t)\widetilde{g}_{jk}dx^j dx^k, \tag{2.3}$$

then we find that

$$R_{tt} = 3\frac{\ddot{a}}{a}, \qquad R_{ij} = -\widetilde{g}_{ij}\left(a\dot{a} + 2\dot{a}^2 + 2k\right), \quad \text{and} \quad R = -6\left(\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2} + \frac{k}{a^2}\right). \tag{2.4}$$

Now, the universe under investigation has a stress-energy tensor which is the sum of those for 'radiation' ($w = \frac{1}{3}$) and 'matter' ($w = 0$) components. Therefore, like above, the 'tt'-Einstein equation is simply

$$3\left(\frac{\dot{a}^2}{a^2} + \frac{k}{a^2}\right) = 8\pi G\left(\rho_m + \rho_r\right), \tag{2.5}$$

which implies

$$\therefore \dot{a}^2 = \frac{8\pi G}{3}a^2\left(\rho_m + \rho_r\right) - k, \tag{2.6}$$

[5] This happens approximately when the small angle expansion of sinh breaks down: when $t \sim \sqrt{\frac{3}{4\Lambda}}$.

[6] Consider, for example a universe with radiation and a $w = -\frac{1}{2}$ quintessence field: such a universe would certainly have the property that radiation is dominant for a negligibly short time in the early/late universe.

where ρ_m and ρ_r are the densities of matter and radiation components of the universe, respectively.

In problem 1 equation (1.8) we derived the relationship $\rho \propto a^{-3(1+w)}$ using only the conservation of energy for a perfect fluid. This therefore most certainly applies for both matter and radiation components of the universe. Expressing the constants of proportionality as

$$\rho_m = \frac{2\beta}{8\pi G}a^{-3} \qquad \text{and} \qquad \rho_r = \frac{3\zeta}{8\pi G}a^{-4}, \tag{2.7}$$

the Einstein equation becomes

$$\dot{a}^2 = \frac{\beta}{a} + \frac{\zeta}{a^2} - k. \tag{2.8}$$

This differential equation is generally solvable in terms of hypergeometric functions, but these are far from enlightening. Rather, we are told to consider the limit that the universe is radiation-dominated for a vanishingly small fraction of its lifetime. This is equivalent to considering 'the age of the universe' to consist almost entirely of that time for which $\beta/a \gg \zeta/a^2$. In this limit, for a closed universe, we have

$$\dot{a}^2 = \frac{\beta}{a} - 1. \tag{2.9}$$

This differential equation can be solved by a clever trick: we know that 1. $a(t)$ has a maximum at $a(t) = \beta$—because then $\dot{a} = 0$—and 2. that $a(t_0) = a(t_f) = 0$. Therefore, we are free to parameterize $a = \frac{\beta}{2}(1 - \cos\eta)$ for some new parameter η. In terms of η, we find that

$$\dot{a}^2 = \frac{1}{1-\cos\eta}(2 - 1 + \cos\eta) = \frac{1+\cos\eta}{1-\cos\eta} = \frac{1-\cos^2\eta}{(1-\cos\eta)^2} = \frac{\sin^2\eta}{(1-\cos\eta)^2}; \tag{2.10}$$

$$\therefore \frac{da}{dt} = \frac{\sin\eta}{1-\cos\eta}. \tag{2.11}$$

Notice that t and η are related by the equation

$$\frac{dt}{d\eta} = \frac{dt}{da}\frac{da}{d\eta} = \frac{1-\cos\eta}{\sin\eta}\frac{\beta}{2}\sin\eta = \frac{\beta}{2}(1-\cos\eta) = a(\eta). \tag{2.12}$$

We are now prepared to determine how many times a photon released at the big bang can encircle the universe before the big crunch. As described above, geodesics which encircle the universe are those for which $r = 1$, $\theta = \pi/2$ in terms of the coordinates of the metric (2.1). Therefore, the condition for a null light ray is simply

$$ds^2 = 0 = -dt^2 + a^2(t)d\varphi^2, \qquad \Longrightarrow \qquad d\varphi = \frac{dt}{a(t)}. \tag{2.13}$$

The total angular distance such a photon can travel during the total time of the universe is then given by

$$\varphi_{tot} = \int d\varphi = \int_{t=0}^{t=t_f} \frac{dt}{a(t)} = \int_0^{2\pi} \frac{dt}{d\eta}\frac{d\eta}{a(\eta)} = \int_0^{2\pi} \frac{a(\eta)}{a(\eta)}d\eta = 2\pi. \tag{2.14}$$

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Problem 3

We are asked to study the asymptotic evolution of a universe filled with matter together with another form of energy[7], termed 'quintessence' with an 'exotic' equation of state $p_Q = w\rho_Q$.

a. We are to determine the equation of state for which quintessence energy density will eventually dominate the universe.

In problem 1, we worked out the dependence of an energy density component in terms of the cosmic scale factor function $a(t)$ and the component's equation of state w (1.8):

$$\rho \propto a(t)^{-3(1+w)}. \tag{3.b.1}$$

Matter, with equation of state $w_m = 0$ is easily seen to evolve according to $\rho_m \propto a(t)^{-3}$. Therefore, any energy component with equation of state $w < 0$ will eventually dominate over matter—as $a(t)$ becomes sufficiently large at late times.

b. We are to solve for $a(t)$ assuming a universe in which quintessence dominates, and find the condition which the equation of state must satisfy so that $a(t)$ remains finite for any finite time.

Assuming that that quintessence is the dominant source of energy density in the universe, we may safely ignore the matter and radiation contributions to Einstein's equation; then, in accordance with (1.6) and (1.8), we find

$$\dot{a}^2(t) = \frac{8\pi G}{3} a^2 \rho_Q \equiv \beta^2 a^{-(1+3w)}(t). \tag{3.c.1}$$

This implies

$$\int a^{(1+3w)/2} da = \beta \int dt. \tag{3.c.2}$$

Now, there are three relevant cases to consider:

- If $w > -1$, this system can be integrated directly: setting $a(0) = 0$[8], we obtain

$$a(t) \propto t^{2/(3(1+w))} \qquad \text{for} \quad w > -1. \tag{3.c.3}$$

- When $w = -1$, we have

$$\int_{a_0}^{a} \frac{da'}{a'} = \beta \int_0^t dt' = \log\left(\frac{a(t)}{a_0}\right), \qquad \Longrightarrow \qquad a(t) = a_0 e^{\beta t} \qquad \text{for} \quad w = -1. \tag{3.c.4}$$

  Notice that this agrees with our results obtained above for a universe with a cosmological constant (for which $w = -1$).

- The (pathological) case of $w < -1$, a bit more care must be taken to evaluate the integral. We find

$$\int_{a_0}^{a} a'^{(1+3w)/2} da' = \beta, t \qquad \Longrightarrow \qquad a(t) \propto \left\{\frac{3(1+w)}{2} t + a_0^{3(1+w)/2}\right\}^{2/(3(1+w))}, \tag{3.c.5}$$

  and bearing in mind that $w < -1$, this has the structure of

$$a(t) \propto \frac{1}{(\eta - \zeta t)^{1/\zeta}} \qquad \text{for} \quad w < -1. \tag{3.c.6}$$

  Clearly, for $w < -1$, $a(t)$ diverges in finite time.

c. We are asked to determine the condition for which the universe has a future horizon.

The null condition on the worldline of a photon travelling in, e.g., the $\hat{x}$-direction is

$$ds^2 = 0 = -dt^2 + a^2(t) dx^2, \qquad \Longrightarrow \qquad dx = \frac{dt}{a(t)}. \tag{3.d.1}$$

[7] The problem explicitly calls this exotic energy density quintessence despite it having nothing at all to do with a model of quintessence. Indeed, there are no models of quintessence for which $w < -1$, but these are considered here anyway.

[8] Actually, this is probably not the boundary conditions we would like to set: because the early universe will be either matter or radiation dominated, it would be more natural to integrate from some some value $a(t_0)$ from whence quintessence dominates. This, however, would not change our primary results.

Therefore, the coordinate distance which a photon can travel is given by

$$x(t=\infty) - x_i = \int_{t_0}^{\infty} \frac{dt}{a(t)}. \tag{3.d.2}$$

It is obvious to anyone with an education including first-semester calculus that this integral is finite only if $a(t) \propto t^{\lambda}$ for $\lambda > 1$—and finiteness of the total distance travelled during an infinite time span indicates the existence of a horizon. Using our work above, we see that there is a horizon if

$$\frac{2}{3(1+w)} > 1, \qquad \Longrightarrow \qquad w < -\frac{1}{3}. \tag{3.d.3}$$

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